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Computing Specter without using Integrals
Part 2

. ***Applications***

Generalities

Now a very intuitive procedure for spectrum computing will be used.
This will be based on the Dirac impulse and the f(t) function graphical representation. We will describe this procedure and also in parallel one analytical procedure, which is useful in the case of computer programming. The most used impulse forms will be analyzed.
The direct and inverse Fourier Transforms are very symmetric and therefore only the spectrum function F(ω) will be calculated. If it is necessary to compute the f(t) function given F(ω) the procedure is the same. The little difference consist of the fact that until f(t) is a real function and *F(ω) generally is a complex function.* In this case we need to decompose *F(ω) in real part* and *imaginary* p*art*, and for each part to compute the Fourier Transform.
Very useful seem to be the f(t) functions classification. In the following the analyzed functions will be of the types :
a) Function of "δ " type.These functions consist only of Dirac impulses and their derivatives
b) –1 L type. The f(t)functions consists of vertical and horizontal lines
-2 Z type The f(t)functions consist of vertical and tilt lines
-3 L Z type The f(t)functions consist of vertical , horizontal and tilt lines
c) P type. Consist of polynomial functions
d) E type The f(t)functions consist of exponential functions
e) M type The f(t)functions consist of modulated functions
f) S type The f(t)functions consist of sinus functions sections
g) V type The f(t)functions consist of above defined functions

The functions spectrum consisting only of Dirac Impulses and its derivatives

In the following the Dirac impulses and Dirac impulse derivatives will be graphically represented by vertical thick lines. Near each line will be a notation that characterizes the concerning impulse (fig A1)
In the figures A1 a), b), c) are represented Dirac impulses and its first and second derivatives that are located in the point t=0

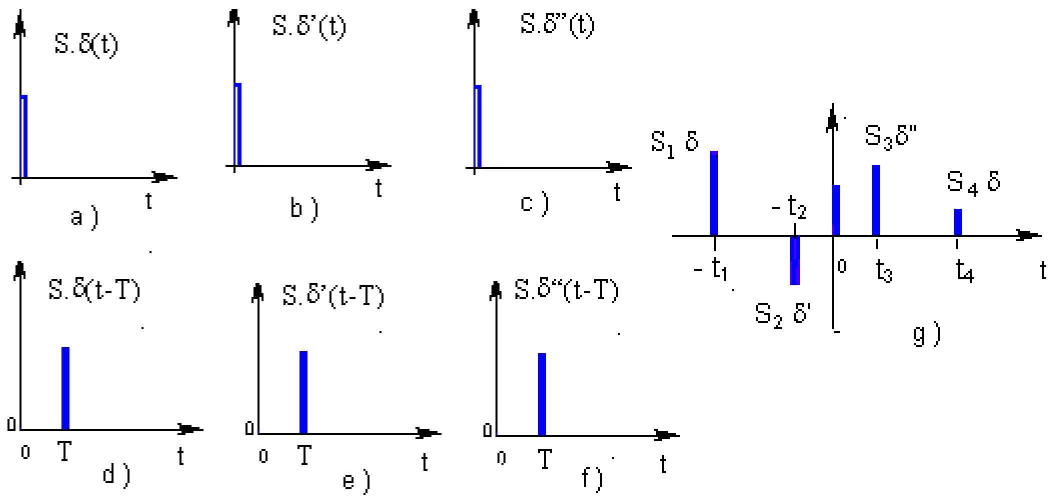

Fig A1 Dirac impulses and its derivatives representation

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The single characteristic parameter is the **surface S**. The spectrum of these functions (based on relation 12) is:

$S.\ \delta(t) \longleftrightarrow S$

$S\ \delta'(t) \longleftrightarrow j\omega S$

$S\ \delta''(t) \longleftrightarrow (j\omega)^2 S$

………………

$S\ \delta^n(t) \longleftrightarrow (j\omega)^n S$ *(A1)*

In the fig A1 d), e), f) are represented *Dirac impulses shifted in the point T*. On the basis of shifted properties (12) results :

$S\ \delta(t-T_0) \longleftrightarrow Sexp(-j\omega T)$

$S\delta'(t-T) \longleftrightarrow \omega Sexp(-j\omega T)$

$S\ \delta''(t-T) \longleftrightarrow (j\omega)^2 Sexp(-j\omega T)$

………………

$S\ \delta^n(t-T) \longleftrightarrow (j\omega)^n Sexp(-j\omega T)$ (A2)

The time function f(t) represented by fig. A1 g is

$f(t) = S_1\delta(t+t_1) - S_2\ \delta'(t+t_2) + S_0\ \delta(t) - S_3\ \delta(t-t_3) + S_4\ \delta''(t-t_4\ )$ *(A3)*

The spectrum function is

$f(t) \longleftrightarrow F(\omega) = S_1 \exp(j\omega t_1) - S_2(j\omega) \exp(j\omega t_2) + S_0 - S_3 \exp(-j\omega t_3) - \omega^2 S_4 \exp(j\omega t)$

The Dirac impulses and its derivatives are simplified. It is now written that these are t functions and are shifted in the concerning points. From the figure results without ambiguities the location points and if are Dirac impulses δ or their derivatives δ' or δ''.

The spectrum function of L type function

a) The general case.

A general L type function is illustrated in fig. A2 a/

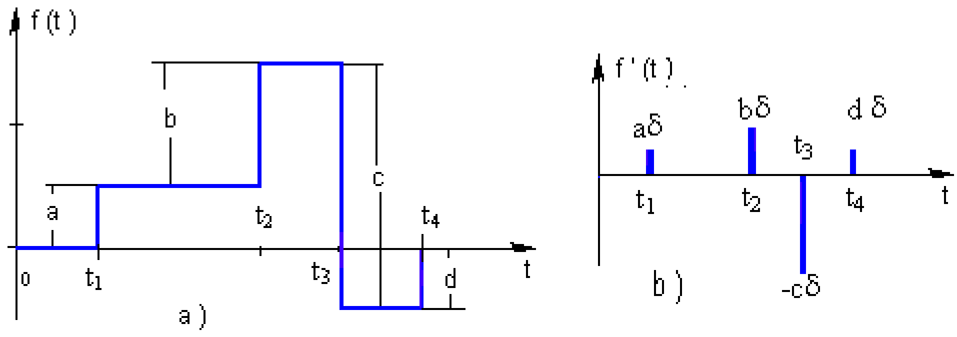


Fig A2 The L type function

The analytical solution

The time function f(t) represented in fig A2, written by using the step functions Φ(t) is:

$$f(t) = a.[\Phi(t-t_1) - \Phi(t-t_2)]. +(a+b)[\Phi(t-t_2) - \Phi(t-t_3)]. -d.[\Phi(t-t_3) - \Phi(t-t_4)] \qquad \text{(A4).}$$

The first derivative is given by:

$$f'(t) = a.\delta(t-t_1) - a.\delta(t-t_2) + (a+b)\,\delta(t-t_2) - (a+b)\,\delta(t-t_3) + d.\,\delta(t-t_4) =$$
$$a.\delta(t-t_1) + b.\delta(t-t_2) - c.[\Phi(t-t_3) - \Phi(t-t_4)] \quad . \qquad (A5)$$

The spectrum function of $f''(t)$ is noted by D(ω) and is given by :

$$. f''(t) \longleftrightarrow a.exp(-j\omega t_1) + b.exp(-j\omega t_2) - c.exp(-j\omega t_3) + d.exp(-j\omega t_4) \qquad \text{(A6)}$$

The f(t) function is the integral of f'(t).Knowing the spectrum function of the f'(t) function we can write directly the spectrum function of f(t) by using the relation (17) and we get:

$$f(t) = int\ (-inf\ to\ t)\ f(t)\ dt \longleftrightarrow F(\omega) = D(\omega)/(j\omega)$$

where $f'(t) \longleftrightarrow .a.exp(-j\omega t_1) + b.exp(-j\omega t_2) - c.exp(-j\omega t_3) + d.exp(-j\omega t_4) = D(\omega)$ (A7)

Solution based on graphical representation

The f(t) function consists on constant line segments defined on different time intervals: t_1.. t_2, t_2…t_3 and t_3.. t_4 The values of the derivatives on these intervals have values equal to zero. .As it was demonstrated in Chapter 4, in the discontinuities points by differentiating result Dirac impulses. The resulting f(t) function consisting only on Dirac impulses is shown in fig A 2b.

On the basis of it we can write direct the spectrum of f(t).

The spectrum function of f(t) results exactly as was shown in the case of analytical solution (relation A7)

The spectrum function of rectangular impulse

In the fig A3.a it is illustrated a rectangular impulse with K amplitude and T width.
The analytical solution for the computing of the spectrum function F(ω)

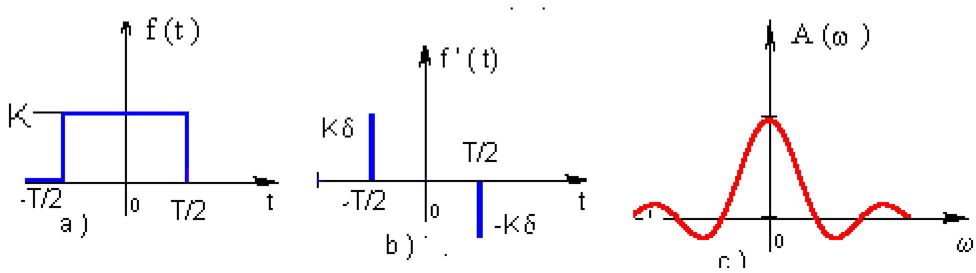


Fig A3

On the basis on given data it is possible to write the f(t) function :

$f(t) = K.[\Phi(t+T/2) - \Phi(t-T/2)]$ (A8).

Differentiating f(t) we get ;

$f'(t) = K.\delta(t+T/2) - K.\delta(t-T/2)$ (A9)

The spectrum function $D(\omega) \longleftrightarrow f'(t)$ is obtained by using the relation A3. We get :

$D(\omega) = K[\exp(j\omega t) - \exp(-j\omega t) = 2.j\,K \sin \omega.T/2$ For $F(\omega)$ (A10)

It results :

$F(\omega) = D(\omega)/(j.\ \omega) = K/(j\omega).2j\sin\omega.T/2 = K/\omega\,(\sin\omega.T/2) = KT(\sin\omega.T/2)/(\omega.T/2) =$
$Kt\,si\,\omega.T/2$ (A11)

and ($\omega.T/2$) is the function

Graphical solution

The f(t) function (figA3 a) is given by a constant K in the time interval –T/2 and +T/2 In this interval the derivative is zero (fig. A3 b) At the discontinuous points result Dirac impulses with the surface equal to K .The spectrum function D(ω) of the f'(t) function can be written directly :

$f'(t) \longleftrightarrow D(\omega) = K[\exp(j\omega t) - \exp(-j\omega t) = 2.j\,K \sin \omega.T/2$ (A12)

The computing of the spectrum function F(ω) is obtained exactly as in the case of analytical solution (A11). In fig A3 c) is illustrated the spectrum function $F(\omega) = A(\omega)$ f(t) being a symmetrical function .The imaginary part B(ω) of the F(ω) is zero and therefore F(ω) = A(ω) A(ω) is given by the known **si** function.

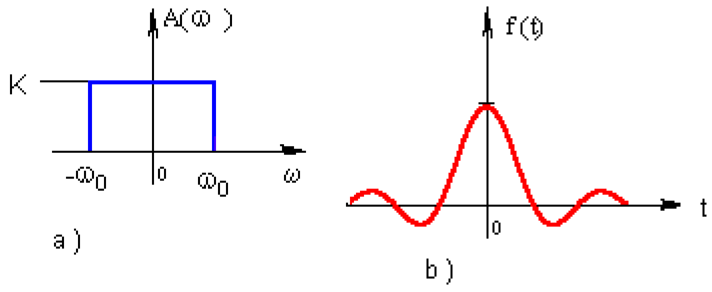
The computing of the f(t) function given the spectrum function F(ω)

This is the case of Fourier inverse transformation. The solution is obtained on the basis of the principle of duality.

F(ω) is illustrated in fig A4 a).Comparing the fig A4a with A3a it is obvious that the solution is obtained if instead of T/2 we write ω_0. Since the difference between the direct and the inverse Fourier transformation is a constant equal with 2π , in this case we have to divide f(t) by 2π.

We get the result:

$$f(t) = (K \sin\omega_0 t)/\pi.t = (K\omega_0\,si\ \omega t)/\pi$$



figA 4 Rectangular impulse in ω domain

This case has a particular importance in electrical domain applications. The f(t) function may be the response of a ideal Low pass filter with the ω_0 pass band limit frequency if at input one Dirac impulse is applied

Limit cases

If the width of f(t) tend to infinite $T \rightarrow \infty$ then f(t) = K (constant) The spectrum function in this case is (infinitely narrow) Dirac impulse with a $2K\pi$ surface (independent of T)

(fig A5)

If the width of f(t) tends to zero **T→ 0** , then f(t) became a Dirac impulse The defining parameter is the surface equal with KT (fig A3).The spectrum function is a constant **equal to S**.

The same values we get if we analyze the fig A4 in the limit cases $\boldsymbol{\omega_0} \rightarrow \infty$ and $\boldsymbol{\omega_0} \rightarrow \mathbf{0}$

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Rectangular impulse pair spectrum

It is proposed to compute the specter functions of the rectangular impulse pairs illustrated in fig 6

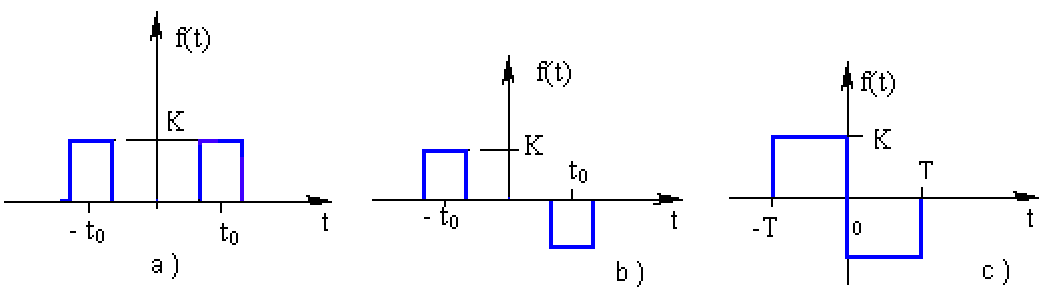


Fig A.6 Rectangular impulse pairs

Case a)

We can use the procedures (analytical or graphical) described above but the simplest is to use the shifting theorem. For example, the case a) the impulse is obtained if we add the impulse f(t) given in fig 3, shifted at right and at left with t_0.

For the resulted specter function is F(ω) we get:

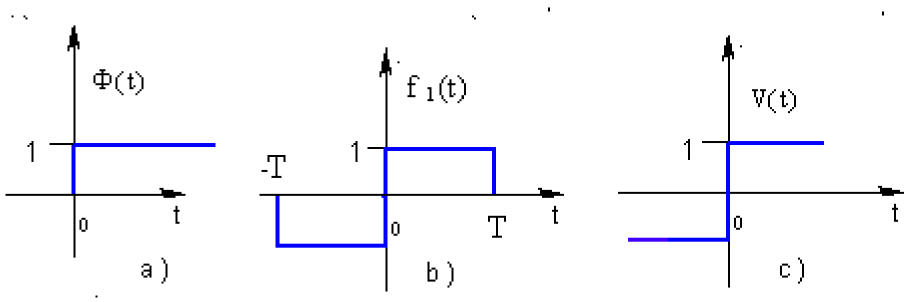
$F_1(\omega) = F(\omega).\exp j\omega t_0 + F(\omega).\exp(-j\omega t_0) = 2F(\omega)\cos\omega t_0 =$
$2KT[si(\omega T/2)].\cos\omega t_0$ (22)

Case b)
In this case the right shifted impulse has a negative amplitude and the resulted spectrum function is:
$F_1(\omega) = F(\omega).\exp j\omega t_0 - F(\omega).\exp j\omega t_0 = 2j\,F(\omega)\sin\omega t_0 =$
$2jKT[si(\omega T/2)].\sin\omega t_0$ (23)

Case c)
This case is a particular case b, where t = T/2. In this case the resulted spectrum function is:
$F_1(\omega) = 2jKT(\sin\omega T/2)/(\omega T/2) . \sin\omega T/2 = 2jKT\sin^2(\omega t/2)/(\omega T/2)$ (24)

The spectrum of the V(t) and Φ (t) functions.

The V(t) function is illustrated in fig A7a This function can be considered as a limited case of the function given in fig A7 b) if T $\rightarrow \infty$ .Then V'(t) consist of a Dirac impulse equal to 2 .δ(t) The spectrum of the 2.δ(t) impulse is 2 and finally we get for the spectrum of the V(t) function;



$V(t) \longleftrightarrow 2/(j\omega)$

Fig A7 V(t) and Φ(t) functions)

The function result as
$\Phi(t) = (V(t)+1)/2$ (A25)
The spectrum of a constant to 1 is $2\pi.\delta(\omega)$ and thus the spectrum of Φ(t) results ;

$$\Phi(t) \longleftrightarrow 1/2(2/(j\omega) + 2\pi\,\delta(\omega)) = 1/(j\omega) + \pi.\delta(\omega)$$

The trapezoidal f (t) function

The trapezoidal function f(t) and the f'(t) and f''(t)derivatives are illustrated in fig A8.

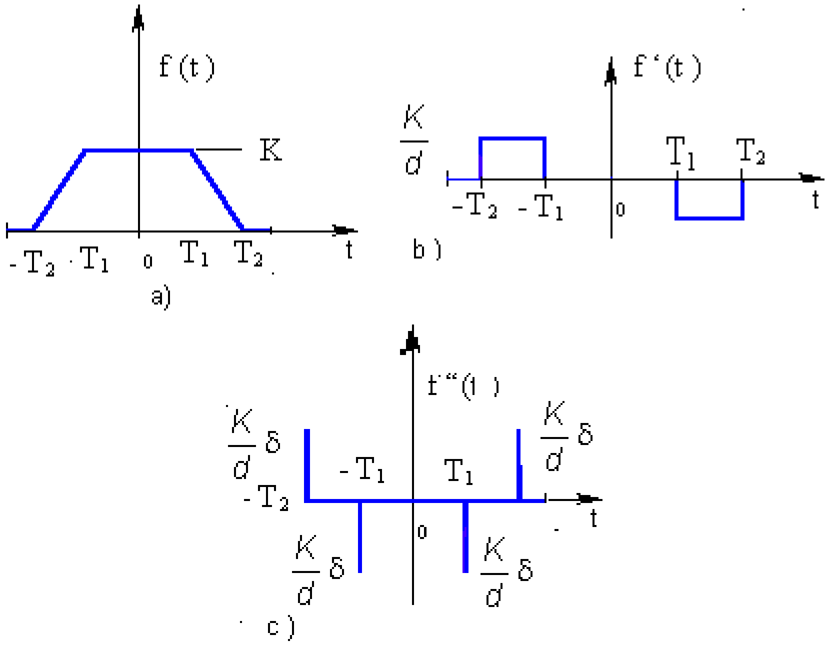


Fig A8 The trapezoidal function and derivatives

We note the line section –T2..-T1 as f1(t) the section –T1..T1 as f2(t)and T1..T2 as f3(t)
For the f(t) function first generalized derivatives ( ) we need only the expressions of the f1'(t), f2'(t) and f3'(t) and not of the f1(t), f2(t) and f3(t). These are:

f1'(t) = K/d   f2'(t)=0 and f3'(t) = -K/d   *where* d = T2-T1

The f(t) function contains only interruption points in –T2, -T1, T1 and T2 and does not contain discontinuity points. Thus the first generalized derivative (34) is

A8 The trapezoidal function

f'(t) = [Φ(t+T2)- Φ(t+T1)].f1'(t) + [Φ(t+T1) - Φ(t-T1)].f2'(t) +[Φ(t-T1) - Φ(t-T2)].f3'(t)=
[Φ(t +T2)- Φ(t+T1)].(K/d) +[Φ(t-T1) - Φ(t-T2)](.-K/d)   *(A26)*

For f''(t) we obtain

*f''(t) =(K/d) (δ(t+T2)- δ(t+T1)) – K/d (δ(t-T1)- δ(t-*T2))   (A27)

The spectrum function D(ω) of the f''(t) results :

D (ω) = K/d(exp(jωT2) +exp(-jωT2) T1) - (exp(jωT1+ exp(jωT1) =
2K/d (cosωT2-cosω T1)   and for

f(t) $\longleftrightarrow$ *F(ω)* = *-D(ω)/*ω^2 = *(-2 K). (cosωT2-cosωT1)/ (*ω^2 *d)*   (A28)

The LZ functions spectrum

One f(t) general LZ function and f'(t) and f''(t) are illustrated in fig A9. The difference between LZ function and the above analyzed functions is that the LZ function contain one vertical section where result a Dirac impulse.

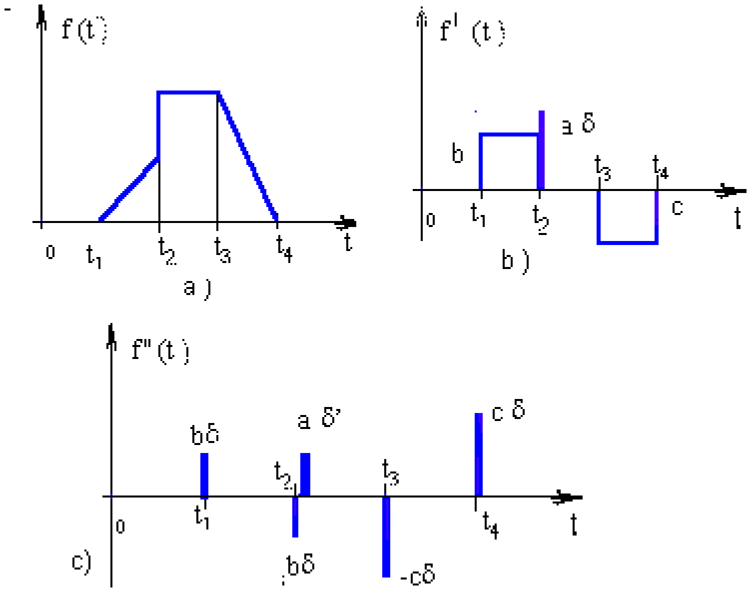


Fig A9 The LZ function

If we note $a = K_1/(t_2 - t_1)$ $b = K_2 - K_1$ $c = K_2/(t_3 - t_4)$

and then using the same reasoning as the previous examples, the first generalized derivative results:

$f'(t) = a([\Phi(t-t_1) - \Phi(t-t_2)] + b\,\delta(t-t_2) - c[\Phi(t-t_3) - \Phi(t-t_4)]$ and

$f''(t) = a.\delta(t-t_1) - a.\delta(t-t_2) + b.\delta'(t-t_2) - c\ t_3) - c\ \delta(t-t_4) - c.\delta(t-t_4)$ (A29)

For the spectrum function D(w) ⟵⟶ f''(t) results;

$D(\omega) = a\ [exp\ (-j\omega t_1) - exp\ (-j\omega t_2) + jb\omega.exp\ (-j\omega t_2) - c.exp\ (-j\omega t_3) + c.exp\ (-j\omega t_4)$ (A30)

And finally the spectrum function

$F(\omega) = D(\omega)/(-\omega^2)$ (A31)

Spectrum of the triangular function

In fig A10 is represented one triangular function f(t) and the derivatives f'(t) and f''(t).

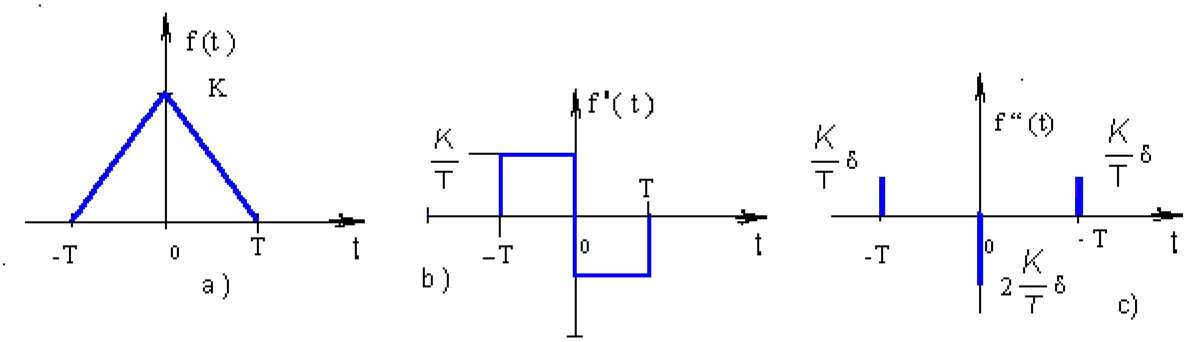


Fig A10 The triangular function

The generalized first derivative f'(t) is ;

$f'(t) = (K/T).([\Phi(t) - \Phi(t-T)] - K\delta(t-T)$

Since in the point T is a discontinuity point a Dirac impulse is resulted with negative surface because in this point is a function decrease. For the second derivative we get :

$$f''(t) = (K/T)\,[.\delta(t) - \delta(t-T)] - .K.\delta'(t-T) \qquad \text{(A32)}$$

For the transformed function of the f''(t) we get :

$$D(\omega) = (K/T)\,[1 - \exp(-j\omega t)] - Kjw\,\exp(j\omega T) \qquad \text{(A34)}$$

and finally

$$F(\omega) = -D(\omega)/\omega^2 \qquad \text{(A35)}$$

Another types of triangular functions

Let a triangle impulse illustrated in fig A11 and we have to compute the spectrum function.

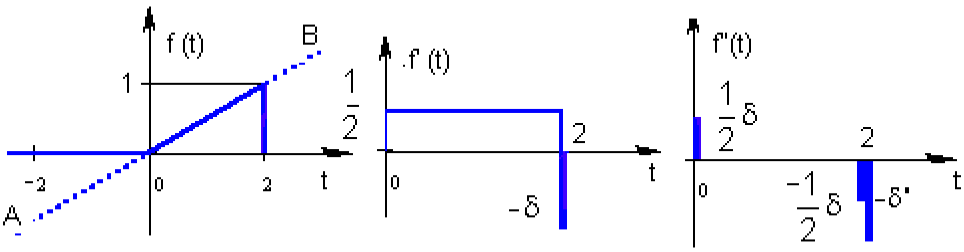


Fig A11

Analytical solution:

The relation of the A-B line is :

$f_{AB}(t) = 0.5.t$

The relation of the f (t) function (limited on the interval { 0; 2} is :

$f(t) = [\Phi(t) - \Phi(t-2)].0.5.t$

The first generalized derivative is:

$f'(t) = 0.5.[\delta(t) - \delta(t-2)] + 0.5.[\Phi(t) - \Phi(t-2)] =$

$= 0.5t.\,\delta(t) - 0.5\,t\,\delta(t-2) + 0.5.[\Phi(t) - \Phi(t-2)]$

Based on the relation (20) we get :

$0.5\,t.\,\delta(t) = 0$ and $0.5t.\,\delta(t-2) = .\,\delta(t-2)$ results :

$f'(t) = -\delta(t-2) + 0.5.[\Phi(t) - \Phi(t-2)]$

The second derivative f” (t) is given by :

$f''(t) = -\delta'(t-2) + 0.5.\,\delta(t) - 0.5.\delta(t-2)$

Based on the relation (20) we get :

$-\delta'(t-2) \longleftrightarrow -j\,\omega\,\exp(j2\,\omega)$ and

$-0.5\,\delta(t-2) \longleftrightarrow 0.5\,\exp(j2\omega)$

$0.5.\delta(t) \longleftrightarrow 0.5$

Finally :

$f''(t) \longleftrightarrow D(\ \ ) = -j.\,\omega\,\exp(2\,j\,\omega) + 0.5 + 0.5\,\exp(j2\omega)$ and

$F(\omega) = D(\omega)/(j\omega)^2 = (-1/\omega^2)\,[0.5 + \exp(j2\omega)(1 - j\omega)]$

b) The graphical solution ;

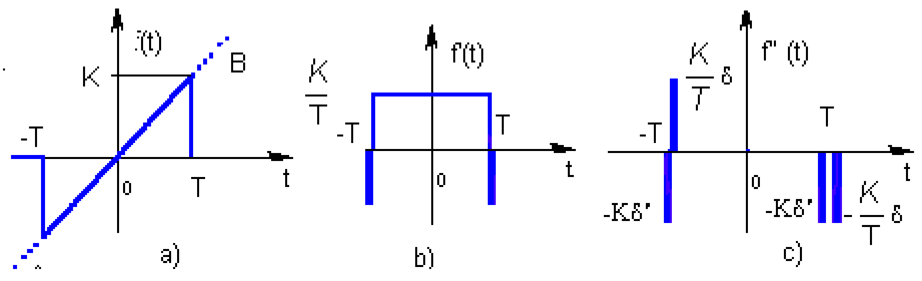
The first f’ (t) derivative and second f “(t) derivative are represented in fig A11 b and A11c

From fig A11c results:

$f''(t) = -\delta'(t-2) + 0.5.\,\delta(t) - 0.5.\delta(t-2)$

The computing of the F (ω) spectrum function is identical with the above computed function.

b) Let's compute the spectrum function of the impulse function represented in in A12



a) b) c

Fig A12

The analytical solution:
The A-B line equation is :

$f_{AB} = (K/T).t$

The relation of the function f (t) limited on the interval {-T; T} is :

$f(t) = \Phi (t + T) - \Phi (t - T).[(K/T).t]$

The first and second derivatives taking in consideration the relations (20) are ;

$f'(t) = (K/T)[\Phi (t + T) - \Phi (t - T) + (K/T).t.[.\delta(t + T) - .\delta(t - T)]$ and

$f''(t) = .(K/T)[\delta(t + T) - .\delta(t - T)] + K[.\delta'(t + T) - .\delta'(t - T)]$

Based on the relation (8) we get

$f''(t) \longleftrightarrow \quad D(\omega) = (k/T)[exp(j\omega T - exp(-j\omega T) + j\omega K (exp(j\omega T) - exp(-j\omega T)]$ and

$F(\omega) = (-1/\omega^2) D(\omega) = -(K/\omega^2)[(1/T)(1- exp(-j\omega T)-j\omega exp(-j\omega T)] =$

$- K. [1 - cos(\omega T) - \omega t \sin (\omega T) + j[\sin (\omega T)- wT cos (\omega T)]/ (T\omega^2)$

The graphical solution

The first and second derivatives of the f(t) function are represented on fig A12 b and A12c

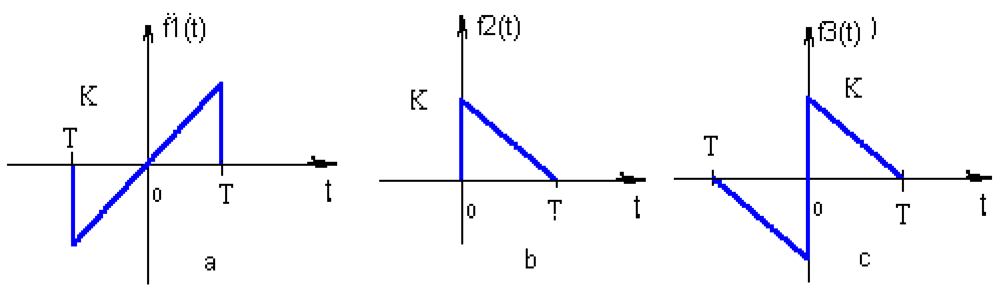
The f"(t) results :

$f''(t) = (K/T) .\delta(t + T) -K.\delta'(t + T)- (K/T).\delta(t - T)- k.\delta'(t - T)$

The spectrum function F(w)results identical with the above analytical solution

b)

In fig A13 we are represented another types of triangular f(t) functions.



FigA13 Different types of triangular functions

It is possible to use the same procedure as in previously cases, but it is simplest to use the shifting and multiplication properties of Fourier transformed functions (6), knowing the F(ω) spectrum function of the f(t) function from fig A11

Case a) (Fig A13 a) The f1(t) in this case is ;

$f1(t) = f(t) – f(-t)$

This expression is twice the odd part of f(t) The Fourier transformation of the odd part is the imaginary part of the spectrum function $F(\omega)$. It results:

$$F1(\omega) = -j.2K.(\sin\omega T-\omega T\cos\omega t)/ (T\omega^2) \qquad (A36)$$

Case b) (FigA13b)
The f2(t) function is obtained from f(t) if the f(t) is shifted at the left with T and then taking f(-t). After shifting with T we get :

$$f2(t) \longleftrightarrow F2(\omega) = F(\omega).\exp (j\omega T) \qquad (A37)$$

Since

$$F(\omega)= -K/(T.\ \omega^2)[1-\exp(-j\omega T)-j\omega T\exp (-j\omega T)] \qquad \text{and then}$$
$$F2(\omega) = K/(T\omega^2)[\ \exp j\omega T-j\omega \exp(-j\omega T] =$$
$$K[1 - \cos\omega T-j(\omega T - \sin\omega T)]/ (T\omega^2) \qquad (A38)$$

Case c)
The f3(t) function from fig A13 c) can be obtained by summing the f2(t) from fig 17b with- f2(-t) But that is twice the odd part of f2(t). The Fourier transformation of the odd part of the f2(t) is the imaginary part of the $F2(\omega)$ It results :

$$F3(\omega) = -j2K(\ \omega T-\sin\omega T)/ (T\omega^2) \qquad (A39)$$