Sofronie Stefanescu
University of Pitesti
Email <sofstef@sunu.rnc.ro

Computing Spectrum without using Integrals
Part I

1. Introduction

Among the many scientific discoveries of the last century, two of them (on the author's opinion) have aided in a spectacular manner the electrical engineering activity. One of them was the **Laplace transformation**, which has spared the engineers of the Differential and Integral equations tedious solving, and another one, which will be described below, was the **Dirac Impulse** use.

The Dirac Impulse use allows the spectrum computing using only simple **algebraic operations** instead of complex and time- and memory-space computing.

This work is principally based on a excellent book written by

Cebe L " **Fourier Integrals es Sor** "

which, regrettably, was not written in English, and therefore is less known by the interested foreign specialists.

2. Fourier Integrals and Properties

Every function can be represented as a sum of sine and cosine waves with different amplitudes and phases, which is called the frequency spectrum (or frequency response) of the function. Such a sum may be infinite.

If the function is represented in that way, i.e., by describing the frequencies and amplitudes, it is called to be depicted in frequency domain. If the function is defined by values at certain time moments (which is a representation most people are used to), it is said to be represented in time domain.

The representation in frequency domain does not contain additional information, in fact it just another way of looking at a function, i.e., a dual representation, but some properties of function are easier understood in frequency domain (which frequencies are present in the function with which amplitudes, etc) other are more obvious in time domain.

The **Fourier transform (FT)** is a mean to switch between time and frequency domain.

There are numerous applications which require the spectrum of a function given as a time-dependent function f (t)

To switch from time domain to the frequency domain and back, two versions of the Fourier transform are needed. The two versions of the Fourier transform are defined by using the **Fourier Integrals**:

$$\mathbf{F(\omega)} = \int_{-\infty}^{\infty} e^{-j\omega t} f(t)dt \qquad \mathbf{(1)}$$

If the spectrum function is known then the corresponding f(t) function is given by :

$$\mathbf{f(t)} = \frac{1}{2.\pi} \int_{-\infty}^{\infty} e^{j\omega t} F(\omega)d\,\omega \qquad (2)$$

(F(ω) is the amplitude density on 1 Hz)

If we have an application on line in which it is required to calculate the spectrum this is a difficult problem because the computation on the basis on (1) is a relative complicate operation for computer programming.

In the following we will describe a method for the determination of all practically used time impulse forms without calculating the Fourier Integral, but using the Dirac Impulse properties.

We will analyze two methods for this computing:

a) f*(t) is given in a analytic form*

b) f(t) is given in a graphical form

For an easy understanding of Dirac integral (function) it is necessary to mention some of the Fourier integral properties.

The linear transformations.

1) If we introduce for direct and **inverse Fourier Transforms** the notations :

$$\mathbf{f(t)} \longleftrightarrow \mathbf{F(\omega)} \qquad \text{and} \qquad \mathbf{F(\omega)} \longleftrightarrow \mathbf{f(t)} \qquad \mathbf{(3)}$$

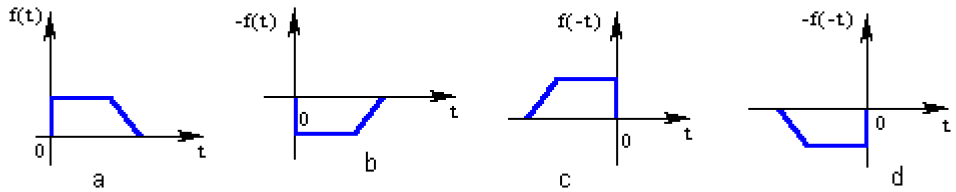
on the basis of superposition principle we get :

$$c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t) \quad \text{----} \quad c_1F_1(\omega) + c_2F_2(\omega) + \ldots\ c_n F_n(\omega) \qquad (4)$$

where $c_1, c_2 .. \ldots c_n$ are constants

The relation (4 results directly from the Fourier integral (1) and (2) definitions. From this relation results that if we break up the f(t) function in some components than the Fourier integral of the original function is the sum of the Fourier integral of the respectively components.
This is valuable for the spectrum function too.

The spectrum functions of f(t) reflected functions.



In fig 1 are represented some simple f(t) transformed functions.
Fig 1

If $\boldsymbol{f(t) \longleftrightarrow F(\omega) = A(\omega) - j\,B(\omega)}$

based on duality and symmetry properties we get; *(5)*

$$-\ f(t) \longleftrightarrow -\ F(\omega) = -A(\omega) + j\,B(\omega) \qquad (6)$$

$$f(-t) \longleftrightarrow F(-\omega) = A(\omega) + j\,B(\omega)$$

$$-\ f(-t) \longleftrightarrow F(-\omega) = -A(\omega) - j\,B(\omega)$$

It is relatively simple to obtain these relations if we break up f(t) in even and odd parts and observe the effect have the above transformations on the Fourier transformation parts.

The contraction (dilatation) on time axis or on frequency axis

If $f(t) \longleftrightarrow F(\omega)$ and *c* is a constant then

$$f(ct) \longleftrightarrow \frac{1}{c}.F\left(\frac{\omega}{c}\right) \text{ and respectively}$$

$$F(c\,\omega \longleftrightarrow \frac{1}{c}.f\left(\frac{t}{c}\right) \qquad (7)$$

We get the above relations if in the Fourier integrals we replace **t or ω** with **c.t** or **c. ω**

From the (7) relation results a very important property and namely that if *f(t) function is c times contracted (c > 1) then F(ω)is c times dilated (extended)on . ω axis, and the amplitude c time is decreasing and the surface stays unchanged. If c <1 then it results the reciprocal effect.*

This property is valid separately for *real part A (ω) and imaginary part B(ω).*

The effects are represented in fig 2

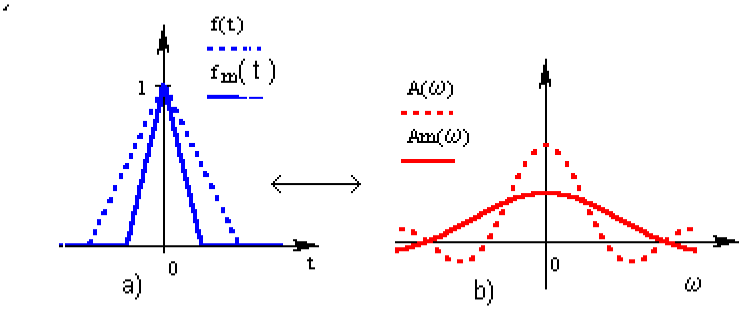


Fig2

The effect of time contraction

It is very important to observe that if the **f (t) function is contracted** then the transformed **F(ω) function is dilated**. The reciprocal is valid too.

. *The spectrum function of f(t) shifted function*.

If f (t) function is shifted with T (T is a constant), then if

$$f(t) \longleftrightarrow F(\omega)$$

$$f(t-T) \longleftrightarrow exp(-j.\omega.t).\ F(\omega) \qquad (8) \quad ....$$

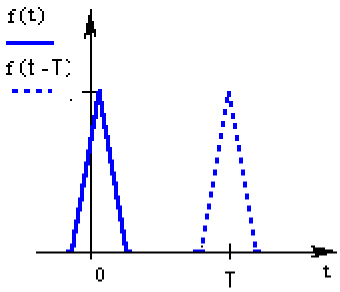


Fig. 3 The translated f(t) function.

The surface theorem

a) Let be the **A(ω)** and **B (ω**) the real and imaginary parts of the spectrum functions which are the Fourier Transform of the f(t) time function.(fig 4) Because f(t) is a real function follow that *B(0) = 0* and then

$$F(0) = A(0) \qquad (9)$$

On the basis of the Fourier integral we get:

$$F(0) = A(0) = \left[\int_{-\infty}^{\infty} e^{-j\omega t} f(t)dt \right]_{(\omega=0)} = \int_{-\infty}^{\infty} f(t)dt = S_t \qquad (10)$$

where $\mathbf{S_t}$ is the surface of the **f(t)** function.

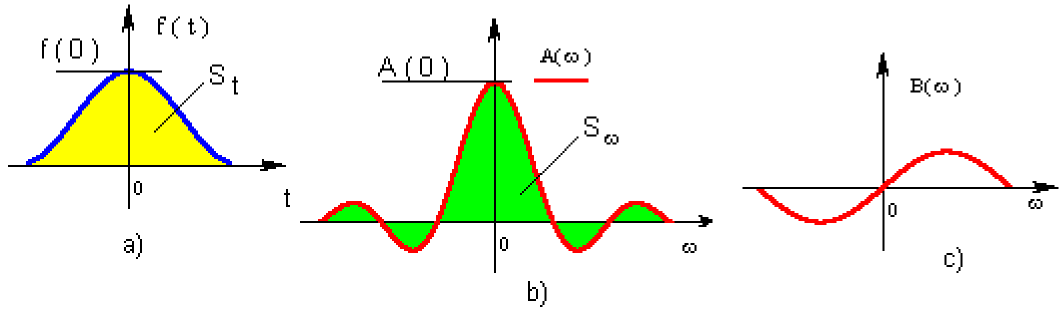


Fig. 4

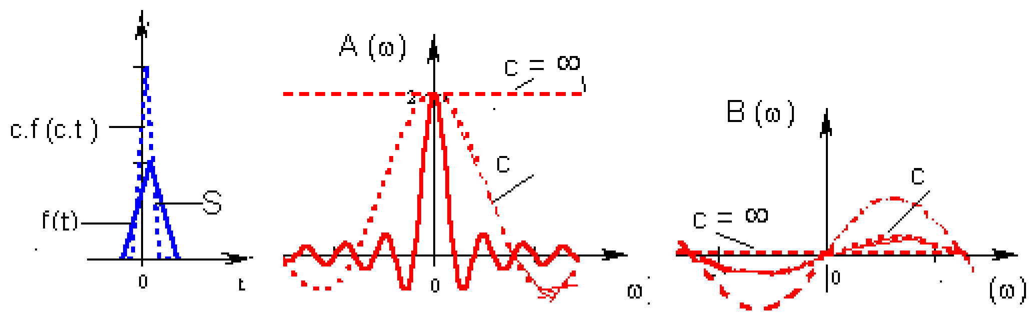
b) Now we compute the **f(0)** value from the spectrum function, also on the basis of Fourier integral

$$\mathbf{f(0)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} F(\omega) d\,\omega\,]_{t=0} = \frac{1}{2\pi} S_\omega \quad . \qquad (11)$$

S_ω is the **A (ω)** function's surface.

3 **The Dirac Impulse**

In the fig. 5 the f(t) function with the S surface is represented. If we introduce the transformation c.f(ct) and c> 1, then the resulting function is c times on the t axis contracted and it has a height c times bigger. In this case the S surface is unchanged.



a) b) c)

Fig.5 The Dirac impulse spectrum

The f(t) function's spectrum is illustrated in the fig 5b and the c.f(ct) function's spectrum for $c \to \infty$ is represented as interrupted curves.

If $c \to \infty$ as is shown in fig.5a the function is infinite narrow, infinite high and has the surface S.

This kind of S_t surface function is defined as **Dirac** impulse. The **unity surface Dirac impulse** is noted as **δ(t)**, and, as it results from fig c (for $c \to \infty$), the Dirac impulse spectrum is a constant equal with the surface S. In conclusion we have

$$\delta(t) \longleftrightarrow 1 \quad \text{and} \quad S.\,\delta(t) \longleftrightarrow S \qquad (12)$$

Since **B (ω) = 0** we conclude that the **δ(t) is a odd function**.

The shifted Dirac impulse spectrum

Usually the Dirac impulse is represented as a thick vertical line and is characterized by a simple parameter, the *surface S*. The shifted Dirac impulse spectrum is shown in fig. 6:

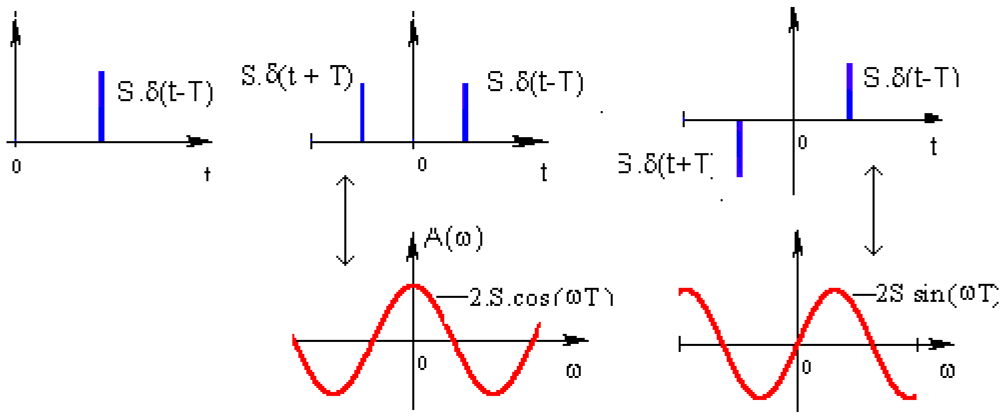


Fig 6 The spectrum of a shifted Dirac impulse and a impulses pair

The spectrum of a Dirac impuls and impulses pair are (8) :

S/2. δ(t-T)) $\longleftrightarrow$ *S/2 .[exp (-jωT)*

S/2.[δ(t-T) + δ (t+T) $\longleftrightarrow$ *S/2 .[exp (-jωT) +exp (jωT) = S.cos ω T*

S/2.[δ(t-T) -δ (t+T)] $\longleftrightarrow$ *S/2 .[exp (-jωT) -exp (jωT) = -j S.sin ω T* *(12)*

The Dirac impulses in the ω domain

On the duality basis we can define a Dirac impulse in the ω domain and we can then compute the *f(t) transformation of the δ (ω) function and of a pair of δ (ω) impulses pair Fig 8*

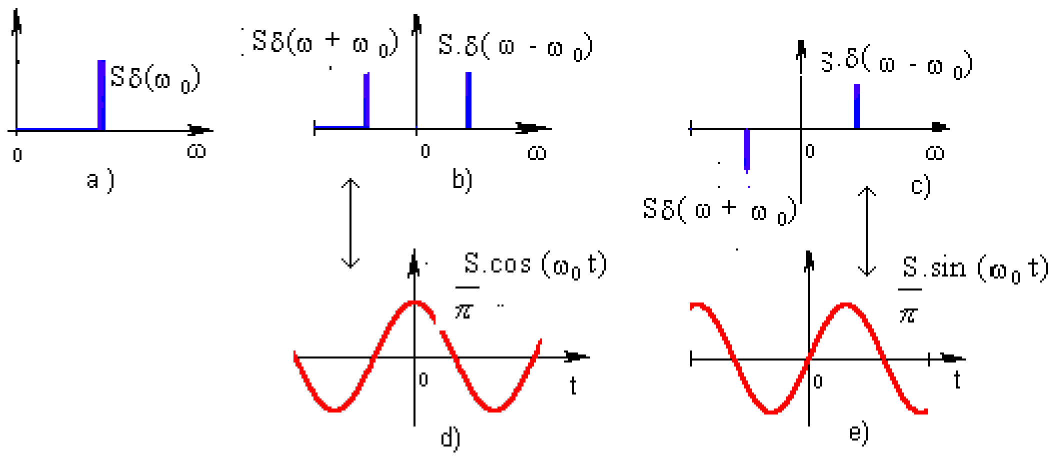


Fig 8 Dirac impulses in ω domain

The Fourier Transform of a ω domain Dirac impulse pair results :

$$f(t) = \frac{1}{2.\pi} \int_{-\infty}^{\infty} \frac{S}{2}.[\ \delta(\omega-\omega_0)+.\ \delta(\omega+\omega_0)]e^{j\,\omega t}\, d\omega =$$

$$\frac{S}{2.\pi}\,\frac{1}{2}\left(e^{j\,\omega_0\,t} + e^{j\,\omega_0\,t}\right) = \frac{S}{2.\pi}\ \cos\,\omega_0\,t$$

(13)

From the fig 8 we read:

$$\cos\,\omega_0 t \longleftrightarrow .\pi\,[\delta\,(\omega\text{-}\omega_0\,) + \delta\,(\omega+\omega_0\,)]$$

$$\sin\,\omega_0 t \longleftrightarrow\ .\ \pi\,[\delta\,(\omega\text{-}\omega_0\,) - \delta\,(\omega+\omega_0\,)]$$

$$S \longleftrightarrow .2\pi.S.\delta\,(\omega\,) \qquad (14)$$

The f'(t) (derivative of f (t)) Fourier integral

If we know the f(t) function's Fourier Transform, we can establish the Fourier transform of the **f '(t) function** on the following reasoning : We take the derivative on the both sides of the Fourier integral :

$$f(t) = \frac{1}{2.\pi}\int_{-\infty}^{\infty} e^{j\omega\,t} F(\omega) d\,\omega$$

$$f\,'(t) = \frac{1}{2.\pi}\int_{-\infty}^{\infty} j\omega t e^{j\omega\,t} F(\omega) d\,\omega \qquad (15)$$

from (15) results :

f ' (t) $\longleftrightarrow$ **jω.F (ω)** and generally for the n order derivative :

$$\mathbf{f^{\,n}\,(t)} \longleftrightarrow \mathbf{(j\omega\,)^{\,n}\,.F(\omega)} \qquad (16)$$

The $\int_{-\infty}^{t} f(t)dt$ ***Fourier transform***

On the basis of above reasoning we obtain

$$\int_{-\infty}^{t} f(t).dt. \longleftrightarrow . \frac{F(\omega)}{j.\omega} \qquad (17)$$

And for the case of n times integral it results for the Fourier transformed function

$$\frac{F(\omega)}{(j.\omega)^{n}} \qquad (18)$$

Finite hight Dirac Impulse

In the fig. 9a an unity surface rectangle is represented with Δ width and 1/Δ heigh

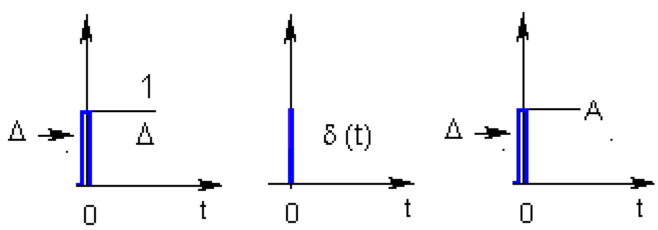


Fig 9

If $\Delta \rightarrow 0$ the impulse become infinitely narrow, infinitely high and with unity surface.
If a function is multiplied with the constant S than their amplitude and surface increases S-times.
It is a difficult question that the Dirac impulse has an infinitely high amplitude. This may be removed if as is shown in fig. 9c the impulse has a A high amplitude and Δ infinite narrow. In this case the impulse surface is

$$A.\Delta \text{ (infinite small).}$$

The resulted Dirac impulse is $A.\ \Delta.\delta(t)$

Dirac impulse shifting

.
If instead of t variable we introduce t-T (T is a positive constant) then the Dirac impulse is shifted in right side with T and becomes **δ(t-T)**.

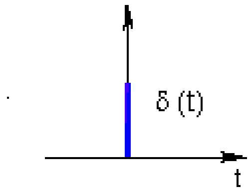


Fig10.Dirac impulse shifting

The multiplication of a function with Dirac impulse.

In fig 11 is shown one f(t) function and a unity Dirac impulse shifted in t_l point If we multiply these two functions it results a Dirac impulse with $f(t_l)$ surface (outside of t_l point the Dirac impulse has zero value). This means that

$$\int_{-\infty}^{\infty} f(t).\ \delta(t-t_l)\, dt = f(t_l)$$

Since

$$f(t)\quad \delta(t-t_l) = f(t_l)\ \delta(t-t_l) \qquad (20)$$

This is a general formula. If

f (t) = exp (j ωt) then

$$\int_{-\infty}^{\infty} e^{j\omega t}.\ \delta(t-t_l)dt = e^{j\omega t_i} \qquad (21)$$

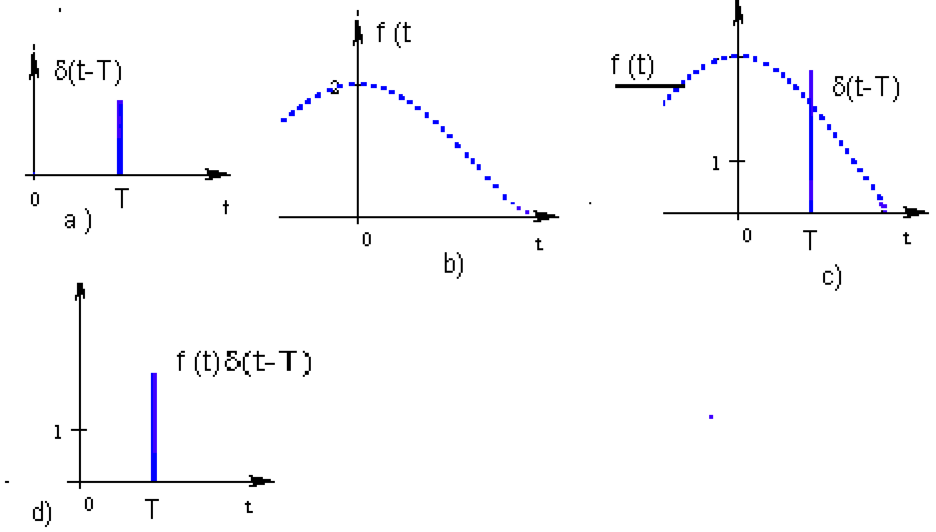


Fig. 11 Multiplication of a Dirac impulse with an f(t) function

4**. Derivative of discontinuous functions**

It is known that there are problems with the derivative of discontinuous functions. For the differential in one discontinuous point the mathematics analysis gives no answer. Due to introduction of Dirac impulse, this problem has a solution.
For the Fourier analysis this is of great importance, because the signal impulses are characterized even with these interruptions (discontinuous) points.

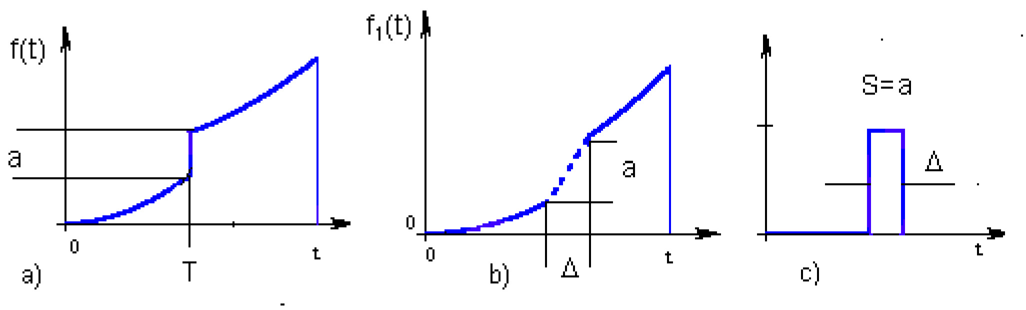


Fig11 Dervative of a discontinous function.

Let f(t) a function with a discontinuous point T (fig 11a) Inside of the vertical line (point T we introduce a linear function with a Δ width and a height. This function can be differentiated without problems and the f ' (t) is shown in fig 11c. The result is a rectangle with a Δ with, a/Δ height and a surface S = a.
If $\Delta \rightarrow 0$, then the linear function becomes a vertical line (fig. 11b) and in the T point the differential as is shown in the fig. 11c represents a **Dirac impulse**. The reciprocal is also true and if we take the integral of the f '(t) function (on the Δ range) we obtain the original function.
The use of Dirac impulse in Fourier analysis will be of a great importance.
The Dirac impulse differential will be noted by **δ'(t)**, the second differential by **δ''(t**) and generally the n order differential by **δ^n(t).**

The δ'(t) and δ^n(t) impulses spectrum.

The obtained properties for normal functions can be applied without difficulties to Dirac functions (impulses) too. If we have

$f(t) \longleftrightarrow F(\omega)$ by applying (16) :

$$f'(t) \longleftrightarrow (j\omega).F(\omega) \qquad (22)$$

If inside of f(t) we introduce δ (t) and $\delta(t) \longleftrightarrow 1$ then

$$\delta'(t) \longleftrightarrow j\omega \qquad \text{and generally}$$

$$\delta''(t) \longleftrightarrow (j\omega)^2$$

$$\cdots\cdots$$

$$\delta^n(t) \longleftrightarrow (j\omega)^n \qquad (23)$$

The step function Φ (t)

The definition of the step function noted as Φ(t) and represented in fig. 12 is

$$\Phi(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \qquad (24)$$

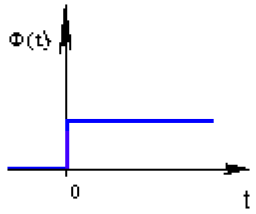


Fig 12 Φ(t) function

The introduction of this function gives us the possibility to describe a function that exists only on a limited interval. If we translate with T_i the above defined Φ (t) function as a normal function we obtain

Φ (t-Ti) (fig13)

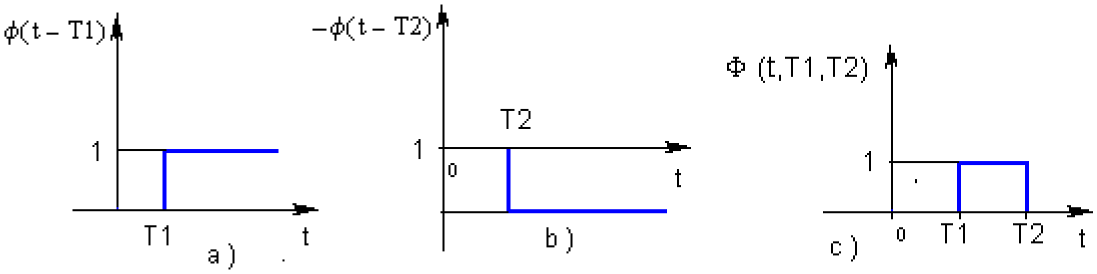


Fig13 The Φ(t-T1) , - Φ (t-T2) and Φ (t-T1)- Φ (t- T2) functions

It is also useful to introduce a new function

$$\boldsymbol{\Phi(t, T1, T2_2) = \Phi(t-T1) - \Phi(t-T2)} \qquad (25)$$

The use of the step functions Φ(t) to describe functions defined on limited intervals.

As a first example let $f(t) = \sin \omega_0 t$. This function is defined on the whole $-\infty$ to $+\infty$ interval (fig 14 a)

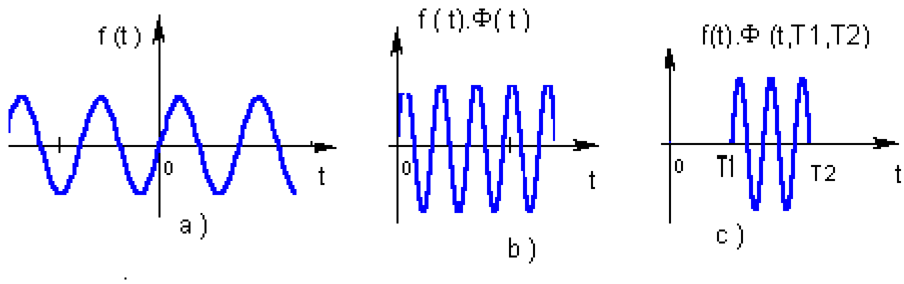


Fig 14 Sinus functions defined on limited interval

If we wish that f(t) be defined only on positive interval, this can be obtained by multiplying f(t) with **Φ(t)** and we obtain:

$$f_1(t) = \Phi(t)\, f(t) = \Phi(t).\sin \omega_0 t \qquad (26)$$

illustrated in (fig 14b)

To describe a function defined on a interval T1 and T2 we have to multiply with Φ(t ,T1,T2)

and it results:

$$f(t) = [\Phi(t-T1) - \Phi(t-T2)].f(t) \qquad (27)$$

In the case of the $\sin \omega_0 t$ function results the curve ilustratedin fig 14c

*The relation between **Φ(t)** and **δ (t)** functions*

Let **Φ(t,T)** the function defined as is shown in fig. 15 a

$$\Phi(t,T) = \begin{cases} 0 & \text{for } t < 0 \\ 1/T & \text{for } 0 < t < T \\ 1 & \text{for } t > T \end{cases} \qquad (28)$$

It is evident that for T. → 0 the Φ(t, T) function become the Φ(t) step function ;

$$\Phi(t,T) = \Phi(t), \quad T. \to 0 \qquad (29)$$

The differential of Φ(t,T) is

$\Phi'(t,T) = \delta(t,T)$ where $\delta(t, T)$ is a rectangle impulse with a T width (fig 15b)

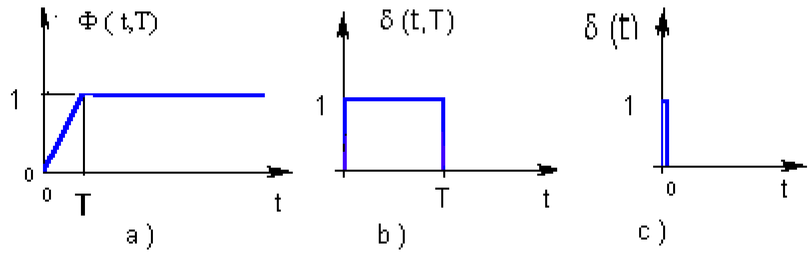


Fig 15

Taking the limit T → 0 we obtain;

$$\Phi'(t,T) = \delta(t,T), \quad T. \to .0 \qquad (30)$$

$$\Phi'(t,T) = \begin{cases} 0 & \text{for } t < 0 \\ 1/T & \text{for } 0 < t < T \\ 1 & \text{for } T < t \end{cases} \qquad (31)$$

It results that the differential Φ'(t) of the Φ(t) is a Dirac impulse δ (t),

$$\mathbf{\Phi'(t) = \delta(t)} \qquad (32)$$

5. The generalized derivative

As was mentioned the mathematical analysis gives no answer to the problem of discontinuous functions derivation. This problem can be solved by using the step function Φ(t) and Dirac impulse δ (t). For exemplification of the mentioned procedure let f(t) (fig. 16) one discontinuous and time limited function. The f(t) function has one t_1 discontinuous point and one t_2 interruption point and is limited between $0..t_3$ time interval.

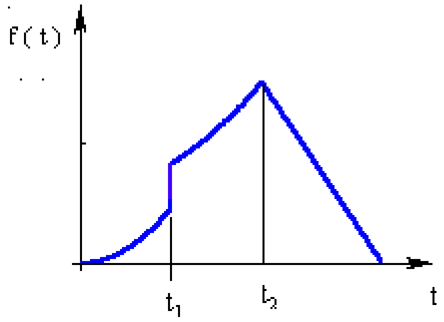


Fig 16 f(t) time limited function with discontinuous and interruption points

The f'(t) is the generalized differential that take in consideration the discontinuous and interruption points and $f^{o}(t)$ the normal differential of f(t).
To describe f(t) and to take in consideration the limited intervals we have to use the step function. For the functions represented in fig 16 it results :

$$f(t) = [\Phi(t) - \Phi(t-t_1)].f_1(t) + [\Phi(t-t_1) - \Phi(t-t_2)].f_2(t) + [\Phi(t-t_2) - \Phi(t-t_3)].f_3(t) \qquad (33)$$

The differential of the f(t) is:

$$f'(t) = [\Phi(t) - \Phi(t-t_1)].f'_1(t) + [\Phi(t-t_1) - \Phi(t-t_2)].f'_2(t) + [\Phi(t-t_2) - \Phi(t-t_3)].f'_3(t) +$$
$$f_1(t).[\delta(t) - \delta(t-t_1)] + f_2(t).[\delta(t-t_1) - \delta(t-t_2) + f_3(t).[(t-t_2) - \delta(t-t_3)] =$$
$$f(t) + f(0). + \delta(t).f(0) + \delta(t-t_1)[f_2(t_1) - f_1(t_1)] + \delta(t-t_2)[f_3(t_2) - f_2(t_2)] - \delta(t-t_3).f_3(t_3)] =$$
$$f^{o}(t) + f_1(0).\ \delta(t) + \Delta f_1.\ \delta(t-t) + \Delta f_2.\ \delta(t-t_2) \qquad . \qquad (34)$$

We have taken in consideration that

$$f^{o}(t) = [\Phi(t) - \Phi(t-t_1)].f'_1(t) + [\Phi(t-t_1) - \Phi(t-t_2)].f'_2(t) + [\Phi(t-t_2) - \Phi(t-t_3)].f'_3(t) \qquad (35)$$

and

$\Delta f_1 = f(t_2) - f(t_1)$ $\Delta f_2 = f(_3) - f(t_2)$ $f_3(t_3)=0$

$f_1(0) = \Delta f_0$ In this case $\Delta f_2 = 0$

Finally it results

$$f'(t) = f^{o}(t) + \Delta f_0\ \delta(t) + \Delta f_1.\ \delta(t-t) + \Delta f_2.\ \delta(t-t_2) \qquad (36)$$

where. $f^{o}(t)$ is given in relation (35)

The second differential of the f(t) function results if we differentiate f '(t) in the same manner and we obtain:

$$f''(t) = [f^{o}(t)]' + \Delta f_0\ \delta'(t) + \Delta f_1.\ \delta'(t-t) + \Delta f_2.\ \delta'(t-t_2) \qquad (37)$$

$[f^{o}(t)]$ is calculated in the same manner as f(t).
It results that the f(t) and f^{o} (t) have the same form, the difference being that in the $f^{o}(t)$ relation instead of of $f_1(t)$, $f_2(t)$ $f_3(t)$ are $f'_1(t)$, $f'_2(t)$ $f'_3(t)$.
The differentiation process continues until the differentiated function contains only Dirac impulses and their differentials.

6.Conclusions *:*

After the first differentiation resulted: $f^{0}(t)$ the differential of the normal function.
In the points t=0 and t= t_1 *resulted two Dirac impulses with the values (surfaces) equal to* $\Delta f_0 = f(0)$ and $\Delta f_1 = f(t_1) - f(t_2$ *which represent the increases (decreases) of the function.*
In the interruption point t_2 *no Dirac impulse resulted.*
No Dirac impulse resulted in the point t_3 *because this is an interruption point (no change in amplitude)*